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On the Kneser property for some parabolic problems

José Valero

Universidad Miguel Hernández, Departamento de Estadística y Matemática Aplicada, Avda. del Ferrocarril, s/n, 03202 Elche, Alicante, Spain

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Abstract

In this paper we consider both a phase-field systems of equations and an abstract differential inclusion for which the uniqueness of the Cauchy problem fails. We prove that the Kneser property holds, that is, that the set of values attained by the solutions at every moment of time is compact and connected. These results are also applied for proving that the global attractors in both cases are connected. An application is given to a reaction–diffusion equation with discontinuous nonlinearity.

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1. Introduction

In this paper we study some parabolic problems like phase-field equations and differential inclusions from the point of view of the connectedness of the set of values attained by the solutions. This problem appears when a differential equation does no posses the property of uniqueness of the Cauchy problem. Several authors have studied this property for ordinary differential equations [5,13], delay differential equations [7,8], reaction–diffusion equations [8,9] or wave equations [1].

In the second section we prove the Kneser property for a phase-field system of equations. The key point in this system is the fact that the nonlinear function f is supposed

E-mail address: jvalero@umh.es (J. Valero).

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only to be continuous (no locally Lipschitz property is assumed to hold). In the third section we consider an abstract parabolic differential inclusion generated by a difference of subdifferentials. An application is given to a reaction–diffusion equation with discontinuous nonlinearity. Finally, in the last section the previous results are applied for proving that the global attractors of these systems are connected. The connectedness of the global attractor is also proved for delay differential equations.

2. Phase-field equations

Consider the phase-field system of equations

$$\begin{cases} \mu\varphi_t - \xi^2\Delta\varphi + f(x,\varphi) = 2u + h_1(x), \\ u_t + \frac{l}{2}\varphi_t = m\Delta u + h_2(x), \quad x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = \varphi|_{\partial\Omega} = 0, \quad t > 0, \\ u|_{t=0} = u_0, \qquad \varphi|_{t=0} = \varphi_0, \quad x \in \Omega, \end{cases} \tag{1}$$

where $\varphi_t = \frac{\partial\varphi}{\partial t}$, $u_t = \frac{\partial u}{\partial t}$, $\Omega \subset \mathbb{R}^3$ is a bounded open subset with smooth boundary, μ, ξ, l, m are positive constants, $h_i \in L^2(\Omega)$, $i = 1, 2$, and the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies:

(F1) $f(x,r)$ is measurable on x, and continuous on r uniformly with respect to x;
(F2) There exists $C \geqslant 0$ such that

$$\begin{aligned} &F(x,r) = \int_0^r f(x,s)\,\mathrm{d}s \geqslant -C, \\ &f(x,r)r - F(x,r) \geqslant -C, \\ &\bigl|f(x,r)\bigr| \leqslant C\bigl(1+|r|^3\bigr). \end{aligned} \tag{2}$$

Let $X = H_0^1(\Omega) \times H_0^1(\Omega)$ be the phase space, $W = L^\infty_{\mathrm{loc}}([0,+\infty); X)$, and let $Q_T = \Omega \times (0,T)$.

Definition 1. The function $\{\varphi, u\} \in W$ is said to be a weak solution of (1) if for any $T > 0$ and any smooth function $\eta(x,t)$, such that $\eta|_{\partial\Omega} = 0$, $\eta(x,T) = 0$ one has

$$\begin{aligned} &-\mu\int_{Q_T} \varphi\eta_t\,\mathrm{d}x\,\mathrm{d}t - \mu\int_\Omega \varphi_0\eta(x,0)\,\mathrm{d}x + \xi^2\int_{Q_T}\varphi_x\eta_x\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} f(x,\varphi,t)\eta\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{Q_T}(2u\eta + h_1\eta)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{3}$$

$$\begin{aligned} &-\int_{Q_T} u\eta_t\,\mathrm{d}x\,\mathrm{d}t - \frac{l}{2}\int_{Q_T}\varphi\eta_t\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega u_0\eta(x,0)\,\mathrm{d}x - \frac{l}{2}\int_\Omega \varphi_0\eta(x,0)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = -m\int_{Q_T} u_x\eta_x\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} h_2\eta\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4}$$

Conditions (F1), (F2) imply that for any $z_0 = (\varphi_0, u_0) \in X$ there exists at least one solution $z = (\varphi, u)$ [6]. Then the set $\mathcal{D}(z_0, T) = \{z\colon\ z(\cdot)$ is a solution on $[0, T]\}$ is well defined (it is the projection onto $[0, T]$ of the set of weak solutions corresponding to z_0). Although the solutions are defined at first in a weak sense, it follows in fact (see [6]) that all weak solutions have good regularity properties:

$$
\begin{aligned}
&u, \varphi \in L^2\big(0, T; H^2(\Omega) \cap H_0^1(\Omega)\big) \cap C\big([0, T]; X\big),\\
&\frac{\mathrm{d}u}{\mathrm{d}t}, \frac{\mathrm{d}\varphi}{\mathrm{d}t} \in L^2\big(0, T; L^2(\Omega)\big), \quad \forall T > 0.
\end{aligned}
$$

Our aim is to prove the Kneser property, that is, that the set $K_t(z_0) = \{z(t)\colon\ z(\cdot) \in \mathcal{D}(z_0, T)\}$ is compact and connected for any $t \in [0, T]$. The compactness was already proved in [6, Lemma 4.8].

Consider the approximation maps f^k, F^k, $k \in \mathbb{N}$, defined by

$$
f^k(x, r) = \begin{cases} f(x, -k), & \text{if } r \leqslant -k,\\ f(x, r), & \text{if } |r| \leqslant k,\\ f(x, k), & \text{if } r \geqslant k, \end{cases}
$$

$$
F^k(x, r) = \int_0^r f^k(x, s)\,\mathrm{d}s = \begin{cases} F(x, -k) + f(x, -k)(r + k), & \text{if } r \leqslant -k,\\ F(x, r), & \text{if } |r| \leqslant k,\\ F(x, k) + f(x, k)(r - k), & \text{if } r \geqslant k \end{cases}
$$

as in [1, Theorem 5.1].

Lemma 2. *There exist constants $C_i \geqslant 0$, $i = 1, \ldots, 4$, such that*

$$
\begin{aligned}
&F^k(x, r) \geqslant -C_1 - C_2\frac{|r|}{k},\\
&f^k(x, r)r - F^k(x, r) \geqslant -C_3,\\
&\big|f^k(x, r)\big| \leqslant C_4\big(1 + |r|^3\big),
\end{aligned}
\tag{5}
$$

for all $r \in \mathbb{R}$, $x \in \Omega$.

Proof. The case $|r| \leqslant k$ is obvious. Let $r \geqslant k$. Then, since $f(x, k) \geqslant -\frac{2C}{k}$ and using (2), we have

$$
\begin{aligned}
F^k(x, r) &= F(x, k) + f(x, k)(r - k) \geqslant -C - 2C\frac{(r - k)}{k} = -C_1 - C_2\frac{|r|}{k},\\
f^k(x, r)r - F^k(x, r) &= f(x, k)r - F(x, k) - f(x, k)(r - k)\\
&= -F(x, k) + f(x, k)k \geqslant -C.
\end{aligned}
$$

The case $r \leqslant -k$ is similar. The last inequality is obvious from (2). $\square$

Let us take a mollifier $\rho_\varepsilon(s)$, $0 < \varepsilon < 1$, and let

$$
f^{k,\varepsilon}(x, r) = \big(\rho_\varepsilon * f^k\big)(x, r) = \int_{\mathbb{R}} \rho_\varepsilon(s) f^k(x, r - s)\,\mathrm{d}s
$$

be the convolution with respect to the second variable. We note that $f^{k,\varepsilon}(x,r) = f^k(x,r)$ if $|r| \geqslant k+1$. Hence, since $r \mapsto f(x,r)$ is continuous uniformly with respect to x, we can choose $\varepsilon_k \to 0$ such that

$$\sup_{r\in\mathbb{R},x\in\Omega} \left| f^{k,\varepsilon_k}(x,r) - f^k(x,r) \right| < \frac{1}{k}.$$

We define then the approximations $f_k = f^{k,\varepsilon_k}$, which are globally Lipschitz with respect to the second variable (the Lipschitz constant C_k depends on k but not on x). We define also the functions $F_k(x,r) = \int_0^r f_k(x,s)\,\mathrm{d}s$.

Lemma 3. *There exist constants $C_i \geqslant 0$, $i = 5,\dots,8$, such that*

$$\begin{aligned} &F_k(x,r) \geqslant -C_5 - C_6\frac{|r|}{k}, \\ &f_k(x,r)r - F_k(x,r) \geqslant -C_7, \\ &\left| f_k(x,r) \right| \leqslant C_8\left(1+|r|^3\right). \end{aligned} \tag{6}$$

Proof. From (5) and the properties of f_k it follows that

$$\begin{aligned} F_k(x,r) &= F^k(x,r) + \int_0^r \left(f_k(x,s) - f^k(x,s) \right) \mathrm{d}s \\ &\geqslant -C_1 - C_2\frac{|r|}{k} - \frac{k+1}{k} \geqslant -C_5 - C_6\frac{|r|}{k}. \end{aligned}$$

If $|r| \leqslant k+1$ we have

$$\begin{aligned} f_k(x,r)r - F_k(x,r) &\geqslant -C_3 + \left(f_k(x,r) - f^k(x,r) \right) r - \left(F_k(x,r) - F^k(x,r) \right) \\ &\geqslant -C_3 - \frac{2}{k}|r| \geqslant -C_3 - 2\frac{k+1}{k} \geqslant -C_7. \end{aligned}$$

If $|r| \geqslant k+1$, then the same results follows because $f_k(x,r) - f^k(x,r) = 0$ and $F_k(x,r) - F^k(x,r) = F_k(x,k+1) - F^k(x,k+1)$.

The last inequality is obvious from the properties of f_k. $\square$

We note also that f_k converges to f in compact subsets of $\mathbb{R}$, and that $f_k(u) \to f(u)$ in $C([0,T],L^2(\Omega))$, for any $u \in C([0,T],H_0^1(\Omega))$ (this is proved in a similar way as in [1, Equality 5.4]).

Theorem 4. *The set $K_t(z_0) = \{z(t)\colon z(\cdot) \in \mathcal{D}(z_0,T)\}$ is connected for any $t \in [0,T]$, $T > 0$ and $z_0 \in X$.*

Proof. Suppose that for some τ the set $K_\tau(z_0)$ is not connected. Then since $K_\tau(z_0)$ is compact, we can find two compact sets A_1, A_2 such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = K_\tau(z_0)$. Let $z_i = (\varphi_i, u_i) \in \mathcal{D}(z_0,T)$, $i = 1,2$, be such that $z_1(\tau) \in U_1$ and $z_2(\tau) \in U_2$, where U_1, U_2 are disjoint open neighborhoods of A_1, A_2. Consider the system

$$\begin{aligned}
&\mu\varphi_t^k-\xi^2\Delta\varphi^k+f_k\bigl(x,\varphi^k\bigr)=2u^k+g_k(x,\lambda,t)+h_1(x)=2u^k+\tilde{h}_{1k}(x,\lambda,t),\\
&u_t^k+\frac{l}{2}\varphi_t^k=m\Delta u^k+h_2(x),\\
&u^k|_{\partial\Omega}=\varphi^k|_{\partial\Omega}=0,\quad t>0,\\
&u^k|_{t=0}=u_0,\qquad \varphi^k|_{t=0}=\varphi_0,\quad x\in\Omega,
\end{aligned}\tag{7}$$

where $\lambda\in[0,1]$ and $g_k(x,\lambda,t)=\lambda(f_k(x,\varphi_1)-f(x,\varphi_1))+(1-\lambda)(f_k(x,\varphi_2)-f(x,\varphi_2))$. We note that $\tilde{h}_{1k}(x,\lambda,t)=g_k(x,\lambda,t)+h_1(x)$ belongs to $C([0,T];L^2(\Omega))$.

We shall obtain some uniform estimates. In [6, Lemma 4.2] the following inequality is obtained

$$\begin{aligned}
&\frac{\mu}{2}\|\varphi_t^k\|^2+\frac{\mu}{2l^2}\|u_t^k\|^2+\frac{\varepsilon_1\xi^2}{8}\|\varphi_x^k\|^2+\frac{m}{l}\|u_x^k\|^2+\varepsilon_1\bigl(f_k\bigl(\varphi^k\bigr),\varphi^k\bigr)\\
&\quad+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\xi^2}{2}\|\varphi_x^k\|^2+\bigl(F_k\bigl(\varphi^k\bigr),1\bigr)+\frac{2}{l}\|u^k\|^2+\frac{\mu m}{l^2}\|u_x^k\|^2+\frac{\varepsilon_1\mu}{2}\|\varphi^k\|^2\right)\\
&\quad\leqslant 2\left(\frac{1}{\mu}+\frac{\varepsilon_1}{\lambda_1\xi^2}\right)\left(\|h_1\|^2+\bigl\|g_k(\lambda,t)\bigr\|^2\right)+\left(\frac{2}{m\lambda_1}+\frac{2\mu}{l^2}\right)\|h_2\|^2+D_1,
\end{aligned}\tag{8}$$

where D_1 is a constant, $\varepsilon_1=\frac{m\xi^2\lambda_1^2}{2l}$ and λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Here, $\|\cdot\|$, $\|\cdot_x\|$ denote the norms in L^2 and H_0^1, respectively. Denoting $Y_k(\lambda,t)=\frac{\xi^2}{2}\|\varphi_x^k\|^2+\frac{\mu m}{l^2}\|u_x^k\|^2+\frac{2}{l}\|u^k\|^2+\frac{\varepsilon_1\mu}{2}\|\varphi^k\|^2+(F_k(\varphi^k),1)$ and using (6) we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}Y_k(\lambda,t)+\delta Y_k(\lambda,t)+\left(\frac{\varepsilon_1\xi^2}{4}-\delta\left(\frac{\xi^2}{2}+\frac{\varepsilon_1\mu}{2\lambda_1}\right)\right)\|\varphi_x^k\|^2\\
&\quad+\left(\frac{m}{l}-\delta\left(\frac{\mu m}{l^2}+\frac{2}{l\lambda_1}\right)\right)\|u_x^k\|^2\\
&\quad\leqslant-(\varepsilon_1-\delta)\bigl(f_k\bigl(\varphi^k\bigr),\varphi^k\bigr)-\delta\bigl(\bigl(f_k\bigl(\varphi^k\bigr),\varphi^k\bigr)-\bigl(F_k\bigl(\varphi^k\bigr),1\bigr)\bigr)\\
&\qquad+D_2\bigl\|g_k(\lambda,t)\bigr\|^2+D_3\\
&\quad\leqslant\left((C_7+C_5)|\Omega|+C_6\frac{\int_\Omega|\varphi^k|\,\mathrm{d}x}{k}\right)(\varepsilon_1-\delta)+\delta D_4+D_2\bigl\|g_k(\lambda,t)\bigr\|^2+D_3\\
&\quad\leqslant D_5+\frac{\varepsilon_1\xi^2}{8}\|\varphi_x^k\|^2+D_2\bigl\|g_k(\lambda,t)\bigr\|,
\end{aligned}$$

for $\delta>0$ small enough, where $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$. We note that the last inequality in (2) and (6) imply that $f_k(\varphi_i(t)),\ f(\varphi_i(t))$ are bounded in $L^2(\Omega)$ uniformly on $t\in[0,T]$, $k\geqslant 1$, and then $\|g_k(\lambda,t)\|\leqslant D_6$, for all $t\in[0,T]$, $k\geqslant 1$, $\lambda\in[0,1]$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}Y_k(\lambda,t)+\delta Y_k(\lambda,t)\leqslant D_7,\quad Y_k(\lambda,t)\leqslant D_8,\tag{9}$$

where the last inequality is uniform in t,k and λ.

With this estimate Galerkin approximations can be used in a standard way to obtain the existence of a solution to (7). For the details see [6, Lemma 4.2] and the references therein. This solution is unique. Indeed, taking the difference of two solutions $(\varphi^1,u^1),(\varphi^2,u^2)$

(we omit now the index k) and using that f_k is globally Lipschitz on the second variable we obtain

$$\mu\|\varphi_t^1-\varphi_t^2\|^2+\frac{\xi^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_x^1-\varphi_x^2\|^2\leqslant\left(\|f_k(\varphi^1)-f_k(\varphi^2)\|+2\|u^1-u^2\|\right)\|\varphi_t^1-\varphi_t^2\|$$
$$\leqslant\frac{\mu}{2}\|\varphi_t^1-\varphi_t^2\|^2+\frac{C_k^2}{\mu}\|\varphi_x^1-\varphi_x^2\|^2+\frac{4}{\mu}\|u^1-u^2\|^2, \tag{10}$$
$$\frac{2\mu}{l^2}\|u_t^1-u_t^2\|^2+\frac{\mu m}{l^2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x^1-u_x^2\|^2$$
$$\leqslant\frac{\mu}{l}\|\varphi_t^1-\varphi_t^2\|\|u_t^1-u_t^2\|\leqslant\frac{\mu}{4}\|\varphi_t^1-\varphi_t^2\|^2+\frac{\mu}{l^2}\|u_t^1-u_t^2\|^2. \tag{11}$$

Summing (10) and (11) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\xi^2}{2}\|\varphi_x^1-\varphi_x^2\|^2+\frac{\mu m}{l^2}\|u_x^1-u_x^2\|^2\right)\leqslant\frac{C_k^2}{\mu}\|\varphi_x^1-\varphi_x^2\|^2+\frac{4}{\mu\lambda_1}\|u_x^1-u_x^2\|^2,$$

from which it follows the unicity by Gronwall lemma.

Now we need to prove that the solution to (7) depends continuously on the parameter λ. We take the difference of the two solutions corresponding to λ_1 and λ_2. Hence, we have

$$\mu\|\varphi_t^1-\varphi_t^2\|^2+\frac{\xi^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_x^1-\varphi_x^2\|^2$$
$$\leqslant\frac{3\mu}{4}\|\varphi_t^1-\varphi_t^2\|^2+\frac{C_k^2}{\mu}\|\varphi_x^1-\varphi_x^2\|^2+\frac{4}{\mu}\|u^1-u^2\|^2$$
$$+\frac{1}{\mu}\|g_k(\lambda_1,t)-g_k(\lambda_2,t)\|^2. \tag{12}$$

The boundedness of $f_k(\varphi_i(t))$, $f(\varphi_i(t))$ in the space $L^2(\Omega)$ uniformly on $t\in[0,T]$ implies that for any $\varepsilon>0$ there exists $\gamma>0$ such that $\|g_k(\lambda_1,t)-g_k(\lambda_2,t)\|^2<\varepsilon$ as soon as $|\lambda_1-\lambda_2|<\gamma$. Then summing (11) and (12) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\xi^2}{2}\|\varphi_x^1-\varphi_x^2\|^2+\frac{\mu m}{l^2}\|u_x^1-u_x^2\|^2\right)\leqslant\frac{C_k^2}{\mu}\|\varphi_x^1-\varphi_x^2\|^2+\frac{4}{\mu\lambda_1}\|u_x^1-u_x^2\|^2+\frac{\varepsilon}{\mu}.$$

Choosing $\nu(k)$ big enough and using Gronwall lemma we have

$$\frac{\xi^2}{2}\|\varphi_x^1-\varphi_x^2\|^2+\frac{\mu m}{l^2}\|u_x^1-u_x^2\|^2\leqslant\frac{\varepsilon}{\mu\nu(k)}\left(\exp(\nu(k)t)-1\right),$$

which proves the continuity with respect to λ.

By the unicity of the solution we have that $\varphi^k=\varphi_2$, $u^k=u_2$, for $\lambda=0$, and $\varphi^k=\varphi_1$, $u^k=u_1$, for $\lambda=1$. Then the continuity property implies the existence of some λ_k such that $z^k(\tau)\notin U_1\cup U_2$.

It follows from (9) that z^k are uniformly bounded in $C([0,T];X)$. Further we multiply the first equation in (7) by φ_t^k, and, after some standard computations, we have

$$\frac{\mu}{2}\|\varphi_t^k\|^2+\frac{\xi^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_x^k\|^2\leqslant\frac{2}{\mu}\|f_k(\varphi^k)\|^2+\frac{4}{\mu}\|u^k\|^2+\frac{2}{\mu}\|g_k\|^2+\frac{2}{\mu}\|h_1\|^2. \tag{13}$$

It follows now from (6), the embedding $H_0^1 \subset L^6$ and the previous estimates for z^k, g_k that the right-hand side of (13) is bounded uniformly in $k \geqslant 1$, $t \in [0,T]$ by a constant D_9. Hence, after integration we have

$$\int_0^T \left\|\varphi_t^k\right\|^2 \mathrm{d}s \leqslant D_{10}. \tag{14}$$

Now, it is easy to obtain also the estimate $\int_0^T \|\Delta\varphi^k\|^2 \,\mathrm{d}s \leqslant D_{11}$. For u_t^k we multiply the second equation in (7) by u_t^k to obtain

$$\frac{1}{2}\left\|u_t^k\right\|^2 + m\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_x^k\right\|^2 \leqslant \|h_2\|^2 + \frac{l^2}{4}\left\|\varphi_t^k\right\|^2. \tag{15}$$

Using (14) we obtain then

$$\int_0^T \left\|u_t^k\right\|^2 \mathrm{d}s \leqslant D_{12}, \qquad \int_0^T \left\|\Delta u^k\right\|^2 \mathrm{d}s \leqslant D_{12}. \tag{16}$$

Hence, we have proved that one can find a subsequence such that $z^k \to z$ weakly in the space $L^2(0,T;(H^2(\Omega)\cap H_0^1(\Omega))^2)$ and such that $z_t^k \to z_t$ weakly in $L^2(0,T;(L^2(\Omega))^2)$. Hence, by the compactness theorem (see [10]), we have that $z^k \to z$ strongly in $L^2(0,T;X)$ and $z^k(x,t) \to z(x,t)$, a.e. in $\Omega \times (0,T)$. Moreover, Ascoli–Arzelà theorem implies that $z^k(t) \to z(t)$ in $L^2(\Omega)$, for any $t \in [0,T]$. A standard argument gives us that $z^k(t) \to z(t)$ weakly in $H_0^1(\Omega)$ for all $t \in [0,T]$.

It follows also that $f_k(\varphi^k) \to f(\varphi)$ weakly in $L^2(0,T;L^2(\Omega))$. Indeed, $f_k(\varphi^k(x,t)) \to f(\varphi(x,t))$, for a.e. $(x,t) \in \Omega \times (0,T)$, and $\|f_k(\varphi^k) - f(\varphi)\|$ is bounded by (6) and the embedding $H_0^1 \subset L^6$. The result is a consequence of [10, Chapter 1, Lemma 1.3].

Let us prove that z is a weak solution. The functions φ^k, u^k satisfy

$$\begin{aligned}
&-\mu\int_{Q_T}\varphi^k\eta_t\,\mathrm{d}x\,\mathrm{d}t - \mu\int_{\Omega}\varphi_0\eta\,\mathrm{d}x + \xi^2\int_{Q_T}\varphi_x^k\eta_x\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} f_k\bigl(x,\varphi^k\bigr)\eta\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{Q_t}\bigl(2u^k\eta + h_1\eta + g_k(\lambda_k,t)\eta\bigr)\,\mathrm{d}x\,\mathrm{d}t,\\
&-\int_{Q_T}u^k\eta_t\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega}u_0\eta\,\mathrm{d}x - \frac{l}{2}\int_{Q_T}\varphi^k\eta_t\,\mathrm{d}x\,\mathrm{d}t - \frac{l}{2}\int_{\Omega}\varphi_0\eta\,\mathrm{d}x\\
&\quad=-m\int_{Q_T}u_x\eta_x\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}h_2\eta\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

for any smooth function η such that $\eta|_{\Omega} = 0$, $\eta(x,T) = 0$, where $\Omega_T = \Omega \times (0,T)$. Passing to the limit as $k \to \infty$ we obtain that z satisfies (3), (4). Repeating the same steps for $2T$, $3T$, etc., and using a standard diagonal argument one can prove that z can be extended to a weak solution, so that $z \in \mathcal{D}(z_0,T)$.

The last step consists in showing that $z^k(\tau) \to z(\tau)$ strongly in $L^2(\Omega)$ (up to subsequence). Integrating over (s,t) in (13) (and operating in a similar way in (1)) the previous estimates imply

$$\begin{aligned}
&\left\|\varphi_x^k(t)\right\|^2 \leqslant \left\|\varphi_x^k(s)\right\|^2 + D_{13}(t-s),\\
&\left\|\varphi_x(t)\right\|^2 \leqslant \left\|\varphi_x(s)\right\|^2 + D_{13}(t-s).
\end{aligned}$$

Define $J_k(t) = \|\varphi_x^k(t)\|^2 - D_{13}t$, $J(t) = \|\varphi_x(t)\|^2 - D_{13}t$. Since these functions are monotone and $J_k(t) \to J(t)$ for a.e. $t \in (0,T)$, we obtain that $J_k(\tau) \to J(\tau)$, and then $\|\varphi_x^k(\tau)\|^2 \to \|\varphi_x(\tau)\|^2$ (see [6, Lemma 4.8] for more details). Hence, $\varphi^k(\tau) \to \varphi(\tau)$ strongly in $H_0^1(\Omega)$.

On the other hand, from (15) and (13) we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_x^k\right\|^2 &\leqslant \frac{1}{m}\left(\|h_2\|^2 + \frac{l^2}{4}\left\|\varphi_t^k\right\|^2\right)\\
&\leqslant \frac{1}{m}\left(\|h_2\|^2 - \frac{l^2\xi^2}{4\mu}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varphi_x^k\right\|^2\right.\\
&\qquad \left. + \frac{2l^2}{\mu^2}\left(\left\|f_k\left(\varphi^k\right)\right\|^2 + \left\|u^k\right\|^2 + \|g_k\|^2 + \|h_1\|^2\right)\right),
\end{aligned}$$

so that

$$\left\|u_x^k(t)\right\|^2 + D_{14}\left\|\varphi_x^k(t)\right\|^2 \leqslant \left\|u_x^k(s)\right\|^2 + D_{14}\left\|\varphi_x^k(s)\right\|^2 + D_{15}(t-s),$$

and we argue as before with the functions $L_k(t) = \|u_x^k(t)\|^2 + D_{14}\|\varphi_x^k(t)\|^2 - D_{15}t$, $L(t) = \|u_x(t)\|^2 + D_{14}\|\varphi_x(t)\|^2 - D_{15}t$.

Finally, since $z^k(\tau) \notin U_1 \cup U_2$, we obtain $z(\tau) \notin U_1 \cup U_2$, which is a contradiction, because $z(\tau) \in K_\tau(z_0)$. $\square$

3. Parabolic equations generated by a difference of subdifferentials

Let us consider now the parabolic equation

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + \partial\psi^1(u) - \partial\psi^2(u) \ni h, & t \in [0,T],\\ u(0) = u_0 \in H, \end{cases} \tag{17}$$

where H is a Hilbert space with the norm $\|\cdot\|$, $h \in H$ and $\partial\psi^i : H \to 2^H$, $i = 1,2$, are the subdifferentials of the proper, convex, lower semicontinuous functions $\psi^i : H \to]-\infty, +\infty]$.

For a (possibly set-valued) map $F : H \to 2^H$ we define the domain of F by $D(F) = \{x \in H{:}\ F(x) \neq \emptyset\}$.

Let us consider the following conditions:

(A1) For all $L < +\infty$ the level set

$$H_L = \left\{u \in H{:}\ \psi^1(u) + \|u\| \leqslant L\right\}$$

is compact.

(A2) $D(\psi^1) \subset D(\partial\psi^2)$, $0 \in D(\psi^1)$, and there exists $C \geqslant 0$, such that

$$\left|\partial\psi^2(u)\right|^+ \leqslant C\big(|u| + 1\big), \quad \text{for all } u \in D\big(\psi^1\big), \tag{18}$$

where $|\partial\psi^2(u)|^+ = \sup_{v\in\partial\psi^2(u)} \|v\|$.

The function $u(\cdot) \in C([0,T];H)$ is called a strong solution of (17) if $u(0) = u_0$, $u(\cdot)$ is absolutely continuous on any compact set of $(0,T)$ and there exist functions $g^i(t)$, $g^i(t) \in \partial\psi^i(u(t))$, a.e. on $(0,T)$, such that

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + g^1(t) - g^2(t) = h, \quad \text{for a.e. } t \in (0,T). \tag{19}$$

Conditions (A1), (A2) imply the existence of at least one strong solution such that $g^2 \in L^2(0,T;H)$ [14, Lemma 1]. Denote by $\mathcal{D}(u_0,T)$ the set of all strong solutions to (17) such that $g^2 \in L^2(0,T;H)$. We shall prove as before that the set $K_t = \{u(t)\colon\ u(\cdot) \in \mathcal{D}(u_0,T)\}$ is compact and connected. The compactness was proved in [14, Lemma 6]. We note also that, since the concatenation of two strong solutions is also a solution (see [14, p. 719]), every $u(\cdot) \in \mathcal{D}(u_0,T)$ can be extended to a function defined on $[0,+\infty)$ such that it is a strong solution for any $T > 0$.

Theorem 5. *The set $K_t(u_0) = \{u(t)\colon\ u(\cdot) \in \mathcal{D}(u_0,T)\}$ is connected for any $t \in [0,T]$, $T > 0$ and $u_0 \in H$.*

Proof. Suppose that for some τ the set $K_\tau(u_0)$ is not connected. Then since $K_\tau(u_0)$ is compact, we can find two compact sets A_1, A_2 such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = K_\tau(z_0)$. Let $u_i \in \mathcal{D}(u_0,T)$, $i = 1,2$, be such that $u_1(\tau) \in U_1$ and $u_2(\tau) \in U_2$, where U_1, U_2 are disjoint open neighborhoods of A_1, A_2.

In this case we shall use another kind of approximations. Let $\partial\psi^2_\lambda$ be the Yosida approximation of $\partial\psi^2$, that is, $\partial\psi^2_\lambda(u) = \frac{1}{\lambda}(u - (1+\lambda\partial\psi^2)^{-1}u)$. This map is defined on the whole space H and is Lipschitz with constant $\frac{2}{\lambda}$ (see [2, p. 73]). Let $u_i^\lambda(t,\gamma)$, $i = 1,2$, $\gamma \in [0,T]$, be defined by $u_i^\lambda(t,\gamma) = u_i(t)$, if $t \in [0,\gamma]$, and let $u_i^\lambda(t,\gamma)$ be the unique strong solution to

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + \partial\psi^1(u) - \partial\psi^2_\lambda(u) \ni h, & t \in [\gamma,T], \\ u(\gamma) = u_i(\gamma), \end{cases} \tag{20}$$

if $t \in [\gamma,T]$ (see [3, Proposition 3.12]). It is clear that $u_1^\lambda(t,0) = u_2^\lambda(t,0)$ and $u_i^\lambda(t,T) = u_i(t)$, $\forall t$.

First we shall prove that the functions $u_i^\lambda(t,\gamma)$ are continuous in γ for any λ, t.

Let $\gamma \downarrow \gamma_0$. If $t \in [0,\gamma_0]$, then $u_i^\lambda(t,\gamma) = u_i(t) = u_i^\lambda(t,\gamma_0)$. On the other hand, we have also that $u_i^\lambda(t,\gamma) = u_i(t)$, for any $t \in [0,\gamma]$. Since $\partial\psi^1$ is monotone and $\partial\psi^2_\lambda$ Lipschitz, making some standard operations in (20) for $t \geqslant \gamma > \gamma_0$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_i^\lambda(t,\gamma) - u_i^\lambda(t,\gamma_0)\right\|^2 \leqslant \frac{2}{\lambda}\left\|u_i^\lambda(t,\gamma) - u_i^\lambda(t,\gamma_0)\right\|^2.$$

It follows by Gronwall lemma that

$$\begin{aligned}&\left\|u_i^\lambda(t,\gamma)-u_i^\lambda(t,\gamma_0)\right\|\\&\quad\leqslant e^{\frac{2}{\lambda}(t-\gamma)}\left\|u_i^\lambda(\gamma,\gamma)-u_i^\lambda(\gamma,\gamma_0)\right\|=e^{\frac{2}{\lambda}(t-\gamma)}\left\|u_i(\gamma)-u_i^\lambda(\gamma,\gamma_0)\right\|\\&\quad\leqslant e^{\frac{2}{\lambda}(t-\gamma)}\left(\left\|u_i(\gamma_0)-u_i^\lambda(\gamma,\gamma_0)\right\|+\left\|u_i(\gamma)-u_i(\gamma_0)\right\|\right)\underset{\gamma\to\gamma_0}{\longrightarrow}0.\end{aligned}$$

Let now $\gamma\uparrow\gamma_0$. If $t\in[0,\gamma]$, then $u_i^\lambda(t,\gamma)=u_i(t)=u_i^\lambda(t,\gamma_0)$. On the other hand, $u_i^\lambda(t,\gamma_0)=u_i(t)$, for any $t\in[0,\gamma_0]$. It is then evident that $u_i^\lambda(t,\gamma)\to u_i^\lambda(t,\gamma_0)$, as $\gamma\to\gamma_0$, for any $t<\gamma_0$. For $t\geqslant\gamma_0$ we obtain (arguing as before) the following:

$$\begin{aligned}&\left\|u_i^\lambda(t,\gamma)-u_i^\lambda(t,\gamma_0)\right\|\\&\quad\leqslant e^{\frac{2}{\lambda}(t-\gamma_0)}\left\|u_i^\lambda(\gamma_0,\gamma)-u_i^\lambda(\gamma_0,\gamma_0)\right\|=e^{\frac{2}{\lambda}(t-\gamma_0)}\left\|u_i^\lambda(\gamma_0,\gamma)-u_i(\gamma_0)\right\|.\end{aligned}$$

The result follows if we prove the continuity for $t=\gamma_0$. We note that $u_i^\lambda(\gamma_0,\gamma)$ is the solution to (20) at $t=\gamma_0$ with initial condition $u(\gamma)=u_i(\gamma)$. Since the function ψ^1 is bounded below by an affine function [2], we have

$$\alpha+(\beta,u)\leqslant\psi^1(u)\leqslant(v,u)+\psi^1(0),\quad\forall v\in\partial\psi^1(u),\ u\in D\left(\partial\psi^1\right),$$

where $\alpha\in\mathbb{R}$, $\beta\in H$. The Lipschitz property implies that $\partial\psi_\lambda^2(u)$ has at most lineal growth. Hence, multiplying by $u_i^\lambda(t,\gamma)$ in (20) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_i^\lambda(t,\gamma)\right\|^2+\alpha+\left(\beta,u_i^\lambda(t,\gamma)\right)-\psi^1(0)\leqslant D_1(\lambda)\left(1+\left\|u_i^\lambda(t,\gamma)\right\|^2\right)$$

and by Gronwall lemma

$$\left\|u_i^\lambda(t,\gamma)\right\|^2\leqslant\left(\left\|u_i(\gamma)\right\|^2+1\right)e^{D_2(\lambda)(t-\gamma)}\leqslant R(\lambda),\quad\forall\gamma\leqslant t\leqslant\gamma_0,$$

where R does not depend on γ. Now, standard arguments involving (20) and (19) give us

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_i^\lambda(t,\gamma)-u_i(t)\right\|^2&\leqslant2\left(\left\|\partial\psi_\lambda^2\left(u_i^\lambda(t,\gamma)\right)\right\|+\left\|g^2(t)\right\|\right)\left(\left\|u_i^\lambda(t,\gamma)\right\|+\left\|u_i(t)\right\|\right)\\&\leqslant D_3(\lambda)\left(1+\left\|g^2(t)\right\|\right),\end{aligned}$$

where g^2 is the function defined in (19) for $u_i(t)$. Since g^2 is integrable, we obtain

$$\left\|u_i^\lambda(\gamma_0,\gamma)-u_i(\gamma_0)\right\|^2\leqslant D_3(\lambda)\left((\gamma_0-\gamma)+\int_\gamma^{\gamma_0}\left\|g^2(t)\right\|\mathrm{d}t\right)<\varepsilon,$$

if $|\gamma_0-\gamma|<\delta(\varepsilon)$. The continuity is proved.

We define now for each fixed λ and t the continuous function

$$\varphi(s,\lambda)(t)=\begin{cases}u_1^\lambda\left(t,\gamma(s)\right),&\text{if }-1\leqslant s\leqslant0,\\u_2^\lambda\left(t,\gamma(s)\right),&\text{if }0\leqslant s\leqslant1,\end{cases}$$

where $\gamma(s)=-Ts$, if $s\leqslant0$, and $\gamma(s)=Ts$, if $s\geqslant0$. It is clear that $\varphi(-1,\lambda)(t)=u_1^\lambda(t,T)=u_1(t)$ and $\varphi(1,\lambda)(t)=u_2^\lambda(t,T)=u_2(t)$. Since $u_i(\tau)\in U_i$ and φ is continuous with respect to s, we have that there exists $s(\lambda)$ such that $\varphi(s(\lambda),\lambda)(\tau)\notin U_1\cup U_2$.

Let us choose some subsequence $\lambda_k\to0$. Let $s_k=s(\lambda_k)$ and denote $u^{\lambda_k}(t,\gamma(s_k))=\varphi(s_k,\lambda_k)(t)$. We shall prove that the sequence $\{u^{\lambda_k}(\cdot,\gamma(s_k))\}$ is convergent in $C([0,T];H)$ (up to a subsequence). We shall need some lemmas.

Lemma 6. *There exists K_1 such that*

$$\left\|u^{\lambda_k}\left(t, \gamma(s_k)\right)\right\| \leqslant K_1, \quad \forall k, \forall t \in [0, T], \tag{21}$$

$$\int_0^T \left|\psi^1\left(u^{\lambda_k}\left(t, \gamma(s_k)\right)\right)\right| \mathrm{d}t \leqslant K_1, \quad \forall k. \tag{22}$$

Proof. We note first that $\|u^{\lambda_k}(t, \gamma(s_k))\| \leqslant \max_{t\in[0,T]}\{\|u_1(t)\|, \|u_2(t)\|\}$, for all $t \in [0, \gamma(s_k)]$. Further, condition (18) and the inequality $\|\partial\psi_\lambda^2(u)\| \leqslant |\partial\psi^2(u)|^+$ (see [3, Proposition 2.6]) imply that $\|\partial\psi_\lambda^2(u)\| \leqslant L_1 + L_2\|u\|$, for all $u \in H$, where L_i do not depend on λ. On the other hand, the monotonicity of the subdifferential map gives us that $(y, u) \geqslant (z, u)$, for any $y \in \partial\psi^1(u),\ z \in \partial\psi^1(0)$. Hence, multiplying (20) by $u^{\lambda_k}(t, \gamma(s_k))$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^{\lambda_k}\left(t, \gamma(s_k)\right)\right\|^2 \leqslant R_1 + R_2\left\|u^{\lambda_k}\left(t, \gamma(s_k)\right)\right\|^2, \quad \text{for a.e. } t \in \left(\gamma(s_k), T\right).$$

Applying Gronwall lemma we obtain (21).

From (20) and the definition of the subdifferential map we have that

$$\begin{aligned}
&\psi^1\left(u^{\lambda_k}\left(t, \gamma(s_k)\right)\right) \\
&\quad\leqslant \psi^1(0) + \left(-\frac{\mathrm{d}u^{\lambda_k}(t, \gamma(s_k))}{\mathrm{d}t} + \partial\psi_{\lambda_k}^2\left(u^{\lambda_k}\left(t, \gamma(s_k)\right)\right) + h, u^{\lambda_k}\left(t, \gamma(s_k)\right)\right),
\end{aligned}$$

for a.e. $t \in (\gamma(s_k), T)$. Hence, inequality (21) implies easily that

$$\int_{\gamma(s_k)}^T \psi^1\left(u^{\lambda_k}\left(t, \gamma(s_k)\right)\right) \mathrm{d}t \leqslant R_3.$$

(We note that $\psi^1(u^{\lambda_k}(t, \gamma(s_k))) \in L^1(\gamma(s_k), T)$ [2, p. 189].) But $\psi^1(u) \geqslant \alpha + (\beta, u)$, $\forall u \in H$, where $\alpha \in \mathbb{R},\ \beta \in H$ [2], so that using (21) again we have $\int_{\gamma(s_k)}^T |\psi^1(u^{\lambda_k}(t, \gamma(s_k)))| \,\mathrm{d}t \leqslant R_4$. Finally, the inequality in $(0, \gamma(s_k))$ follows from the fact $\psi^1(u_i) \in L^1(0, T)$ (see [14, Lemma 3]). □

Lemma 7. *There exists K_2 such that*

$$t\psi^1\left(u^{\lambda_k}\left(t, \gamma(s_k)\right)\right) \leqslant K_2, \quad \forall t \in (0, T],\ \forall k, \tag{23}$$

$$\int_0^T t\left\|\frac{\mathrm{d}u^{\lambda_k}(t, \gamma(s_k))}{\mathrm{d}t}\right\|^2 \mathrm{d}t \leqslant K_2, \quad \forall k. \tag{24}$$

Proof. We note first that (23), (24) are valid for the functions u_i, where the estimate is uniform for $u_0 \in B$, a bounded set [14, Lemma 4].

Consider first the case where $\gamma(s_k) \neq 0$, $\forall k$. It is sufficient in fact if this property holds for some subsequence. We note that in this case $\psi^1(u_i(t))$ are absolutely continuous functions on $[\gamma(s_k), T]$ [14, Lemma 3] and then $u_i(\gamma(s_k)) \in D(\psi^1)$, $\forall k$. We note also that $\partial\psi_\lambda^2(u_i^{\lambda_k}) \in L^2(\gamma(s_k), T; H)$ [3, p. 107] and then $\frac{\mathrm{d}u_i^{\lambda_k}}{\mathrm{d}t} \in L^2(\gamma(s_k), T; H)$ [2, p. 189]. Hence, for $g(t) = \partial\psi_\lambda^2(u_i^{\lambda_k}(t, \gamma(s_k))) + h - \frac{\mathrm{d}u_i^{\lambda_k}(t,\gamma(s_k))}{\mathrm{d}t} \in \partial\psi^1(u_i^{\lambda_k}(t, \gamma(s_k)))$, we have $(g(t), \frac{\mathrm{d}u_i^{\lambda_k}}{\mathrm{d}t}) = \frac{\mathrm{d}}{\mathrm{d}t}\psi^1(u_i^{\lambda_k}(t, \gamma(s_k)))$, a.e. on $(\gamma(s_k), T)$ (see [2, p. 189]). Now, multiplying (20) by $t\frac{\mathrm{d}u^{\lambda_k}}{\mathrm{d}t}$, integrating by parts and using (21) and $\|\partial\psi_\lambda^2(u)\| \leqslant |\partial\psi^2(u)|^+ \leqslant L_1 + L_2\|u\|$ we have

$$\frac{1}{2}\int_{\gamma(s_k)}^{t} s\left\|\frac{\mathrm{d}u^{\lambda_k}}{\mathrm{d}t}\right\|^2 \mathrm{d}s + t\psi^1\big(u^{\lambda_k}\big(t, \gamma(s_k)\big)\big)$$
$$\leqslant \gamma(s_k)\psi^1\big(u_i\big(\gamma(s_k)\big)\big) + R_4 + \int_{\gamma(s_k)}^{t} \psi^1\big(u^{\lambda_k}\big)\,\mathrm{d}s. \tag{25}$$

From the cited properties of u_i at the beginning of the proof we have $\gamma(s_k)\psi^1(u_i(\gamma(s_k))) \leqslant R_5, \forall k$. Hence, (22) and $u^{\lambda_k}(t, \gamma(s_k)) = u_i(t),\ \forall t \in [0, \gamma(s_k)]$, imply that $\int_0^T s\|\frac{\mathrm{d}u^{\lambda_k}}{\mathrm{d}t}\|^2\,\mathrm{d}s \leqslant R_6$, $t\psi^1(u^{\lambda_k}(t, \gamma(s_k))) \leqslant R_6$ on $(0, T]$.

Let now $\gamma(s_k) = 0$, $\forall k$. Let us consider a sequence u_0^N such that $u_0^N \underset{N\to\infty}{\longrightarrow} u_0$ in H, $u_0^N \in D(\psi^1)$. From (25) with $\gamma(s_k) = 0$ we obtain that (23), (24) hold for any N and the constant K_2 do not depended on N. It is well known that $u_{i,N}^{\lambda_k}(t, 0) \underset{N\to\infty}{\longrightarrow} u_i^{\lambda_k}(t, 0)$ in $C([0, T]; H)$ (see [3, Proposition 3.14]). Hence, the lower semicontinuity of ψ^1 implies that

$$t\psi^1\big(u^{\lambda_k}(t, 0)\big) \leqslant \liminf_{N\to\infty} t\psi^1\big(u_N^{\lambda_k}(t, 0)\big) \leqslant K_2, \quad \text{for } t \in (0, T].$$

Finally, (24) follows also by passing to the limit. □

Now, repeating exactly the same arguments as in [12, pp. 592–594] we obtain that a subsequence u^{λ_k} converges to some strong solution $u_{\gamma_0}(\cdot)$ in $C([\gamma_0, T]; H)$, where $\gamma(s_k) \to \gamma_0$. On the other hand, $u^{\lambda_k}(t, \gamma(s_k)) = u_i(t)$, where either $i = 1$ or $i = 2$, for all k, so that u^{λ_k} converges to the function

$$u(t) = \begin{cases} u_i(t), & \text{if } t \in [0, \gamma_0], \\ u_{\gamma_0}(t), & \text{if } t \in [\gamma_0, T]. \end{cases}$$

We note that $u_{\gamma_0}(\gamma_0) = u_i(\gamma_0)$. Also, if $\gamma_0 > 0$, then $u_i(\gamma_0) \in D(\psi^1)$ (see [14, p. 719]), so that $\frac{\mathrm{d}u_{\gamma_0}}{\mathrm{d}t} \in L^2(\gamma_0, T; H)$ [2, p. 189] and u_{γ_0} is absolutely continuous on $[\gamma_0, T]$. Hence, it follows that $u(\cdot)$ is a strong solution (the fact that u satisfies (19) is proved in the same way as in [14, p. 715]). It follows also from the limiting process that the function g^2 corresponding to $u(\cdot)$ belongs to $L^2(0, T; H)$. But then $u(\tau) \notin U_1 \cup U_2$ and $u \in \mathcal{D}(T, u_0)$, which is a contradiction. □

We shall consider now the application of the proved theorem to the following reaction–diffusion equation with discontinuous nonlinearities:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + f_1(u) - f_2(u) \ni h, \quad \text{in } \Omega \times (0,\infty), \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{26}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary $\partial\Omega$, $h \in L^2(\Omega)$ and $f_i : \mathbb{R} \to 2^{\mathbb{R}}$, $i = 1, 2$, are maximal monotone maps with $D(f_i) = \mathbb{R}$. We suppose that there exist $K_1, K_2 \geqslant 0$ such that

$$\left|f_2(s)\right|^+ \leqslant K_1 + K_2|s|, \tag{27}$$

where $|f_2(s)|^+ = \sup_{y\in f_2(s)} |y|$. We put $H = L^2(\Omega)$ and

$$\partial\psi^1(u) = \left\{y \in H:\ y(x) \in -\Delta u(x) + f_1\big(u(x)\big),\ \text{a.e. on } \Omega\right\},$$
$$\partial\psi^2(u) = \left\{y \in H:\ y(x) \in f_2\big(u(x)\big),\ \text{a.e. on } \Omega\right\},$$

for suitable ψ^i (see [14] for more details). We can easily check that (27) implies (18). The other conditions in (A1), (A2) are proved in [14, p. 733].

Hence, Theorem 5 implies that the set $K_t(u_0)$ is connected. In [14, Examples 2–4] some models of physical interest (as a model of combustion in porous media and a model of conduction of electrical impulses in nerve axons) are studied as particular cases of (26).

4. Connexion of the global attractor

In [6,14] it is proved the existence of a global attractor for the equations studied in the previous sections. Our aim now is to prove that in both cases the global attractor is connected.

Let us recall briefly the main definitions of the theory of attractors. Let X be a complete metric space with the metric ρ. The multivalued map $G : \mathbb{R}^+ \times X \to P(X)$ ($P(X) = \{A \subset X:\ A \text{ is nonempty}\}$) is called a strict multivalued semiflow if $G(0,\cdot) = \mathrm{Id}$ and $G(t_1 + t_2, x) = G(t_1, G(t_2, x))$, for any $x \in X$, $t_i \in \mathbb{R}^+$. The set $\mathcal{A}$ is said to be a global attractor if it attracts any bounded set $B \subset X$, that is,

$$\operatorname{dist}\big(G(t,B),\mathcal{A}\big) \to 0, \quad \text{as } t \to +\infty,$$

where $\operatorname{dist}(C,D) = \sup_{c\in C} \inf_{d\in D} \rho(c,d)$, and $\mathcal{A} \subset G(t,\mathcal{A})$, for all $t \geqslant 0$. We note that the excess of the set C over D, denoted by $\operatorname{dist}(C,D)$, is not a distance in fact.

A trajectory of G is a function $x(\cdot) : \mathbb{R}^+ \to X$ such that $x(t+\tau) \in G(t, x(\tau))$, $\forall t, \tau \in \mathbb{R}_+$. The semiflow G is called time-continuous if

$$G(t,x_0) = \bigcup \big\{x(t):\ x(\cdot) \text{ is a trajectory and}$$
$$x(\cdot) \in C(\mathbb{R}^+; X),\ x(0) = x_0\big\}, \quad \forall x_0 \in X.$$

Theorem 8 [11, Theorem 5]. *Let G be a strict time-continuous m-semiflow with closed and connected values. Let the map $x \mapsto G(t,x)$ be upper semicontinuous for any $t \geqslant 0$. If G has a global attractor $\mathcal{A}$ and the phase space X is connected, then $\mathcal{A}$ is connected.*

We note that the semiflows corresponding to (1) and (17) are defined in [6,14] by taking the union of all solutions, which are continuous functions defined on $\mathbb{R}^+$ and are proved to be trajectories, so that these semiflows are time-continuous. From the results in [6] it follows that the system (1) generates a strict time-continuous multivalued semiflows which has a compact global attractor. The same result is proved in [14, Theorem 3] for Eq. (17) under the following additional dissipative condition: there exist $\delta > 0$, $M > 0$ such that $(y_1 - y_2 - h, u) \geqslant \delta$, for all $u \in D(\partial\psi^1)$, $|u| > M$, $y_i \in \partial\psi^i(u)$. The only property in Theorem 8 left to prove in those papers is the fact that the semiflow has connected values. This follows now from Theorems 4 and 5. Hence, we have:

Theorem 9. *The global attractors corresponding to Eqs.* (1) *and* (17) *are both connected.*

Remark 10. For Eq. (26) the same result is valid under the following additional dissipative condition: there exist $M \geqslant 0$, $\varepsilon > 0$, such that $(y_1 - y_2)s \geqslant (-\lambda_1 + \varepsilon)s^2 - M$, for all $y_i \in f_i(s)$, $i = 1, 2$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Finally, let us consider the functional differential equation

$$x'(t) = f(x_t), \quad x_0 = \psi \in \mathcal{C}, \tag{28}$$

where $\mathcal{C} = \mathcal{C}([-h, 0], \mathbb{R}^n)$ endowed with the supremum norm $\|\cdot\|$, $h > 0$, $x_t(\theta) = x(t + \theta)$, and $f : \mathcal{C} \to \mathbb{R}^n$ is a continuous function such that $|f(x)| \leqslant C_1 + C_1\|x\|$, $\forall x \in \mathcal{C}$. Then it is well known [8, Theorem 4] that the set $\mathcal{D}(x_0, T) = \{x\colon x(\cdot) \text{ is a solution on } [0, T]\}$ is well defined and satisfies the Kneser property.

In [4] it is defined for such equations a strict multivalued time-continuous semiflow and under some conditions on f it is proved the existence of a compact global attractor. Now it follows from Theorem 8 that the global attractor is connected if the function f has at most linear growth.

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